Solving Equations

"Solving" an equation just means finding all the possible values for the unknown variable in the equation that will make the equation true. Often there's only one solution, but sometimes there's more than one possible value for a variable and we want to find all possible values that it could take on to make the equation true. The way we do it is by assuming an equation is a true statement, like "$2x + 3 = 5$" is true, and then rearrange things and use some algebra to write another true statement, like "$2x = 2$," and rearrange things some more to get another true statement, like "$x = 1$" and then the solution is obvious. If x is 1, then that equation is true. If everything works out peachy, then the value of x that makes the last equation true is also a value for x that makes the original equation true (everything doesn't always work out peachy, but it usually does).

The Golden Rules of Equation Solving are about making sure you don't take something true (a good equation) and accidentally turn it into something untrue (a bad equation). So what are the Golden Rules?

> **The Golden Rules of Equation Solving**
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> Do whatever thou wisheth to an equation, as long as you
>
> (1) do unto one side of the equation exactly the same as thou doest unto the other, or
>
> (2) replaceth a thing by something equal to it.

These are actually common sense. The first rule is like saying "If $x = 13$, we know that $x + 7 = 13 + 7$." That makes sense. Don't limit yourself to the simpler arithmetic operations, though. As long as you do *exactly* the same thing to both sides, you're *usually* in good shape (however, it's always crucial to check your answers!). Some more exotic possibilities you might want to consider at some point:

- If $A = B$, then $A^2 = B^2$.
- If $A = B$, then $5^A = 5^B$ (of course, there's nothing special about 5 in this case, you could use any other number instead).
- If $A = B$, then $1/A = 1/B$.
- More possibilities await...

The second rule is also common sense. It's really saying something like "If $x = y$, and $x + 28 = \sqrt{375}$, then $y + 28 = \sqrt{375}$." Again, that seems obvious. If things are equal, swap them out however you want.

The Golden Rules of Equation Solving will help you avoid writing down things that aren't true. However, when we're solving equations, we also want to find *all* of the solutions, not just some of them. Here's a silly example of how we could "lose" solutions when solving an equation:

To solve the equation $x^3 - 4x + 18 = \frac{4}{x-8}$, we could try to simplify things by multiplying both sides by 0:

$$0 \cdot (x^3 - 4x + 18) = (\frac{4}{x-8}) \cdot 0$$
$$0 = 0$$

We followed the Golden Rules, and we ended up with something true. But that didn't help us figure out what x is. So, we have to be careful and try to not make moves that cause us to lose solutions. Here's a more subtle example:

I'm thinking of a number x. The reciprocal[1] of x is the same as x. What is x?

If we express this problem in equation form, we would get $x = \frac{1}{x}$. Let's try to solve it, following the Golden Rules.

[1]Remember: the "reciprocal" of a number is 1 divided by that number. The reciprocal of 5 is 1/5. The reciprocal of 3/4 is $1/(3/4) = 4/3$.

$$
\begin{aligned}
x &= \frac{1}{x} \\
x^2 &= 1 \qquad \text{(multiply both sides by } x\text{)} \\
x &= 1 \qquad \text{(take the square root of both sides)}
\end{aligned}
$$

Since we followed the Golden Rules, we know that $x = 1$ should be a true equation, and if we look back at the original equation, $x = 1$ is a solution. But there's another solution! We could also say $x = -1$, and that equation would be true. What happened? (Think about it, try to figure it out.)

Let's see another common way in which we can "lose" solutions.

My daughter has two friends. One is three years younger than her, and the other is two years older than her. The product of her friends' ages is 104. How old is my daughter?

We can set up an equation to represent this situation. If my daughter's age is d, then her friends ages must be $d - 3$ (three years younger) and $d + 2$ (two years older). The product of these two ages is 104, so:

$$(d - 3)(d + 2) = 104.$$

Let's go through one possible way someone might try to solve this equation.

$$
\begin{aligned}
(d - 3)(d + 2) &= 104 \\
d^2 - d - 6 &= 104 \qquad \text{(multiply out the left-hand-side using FOIL)} \\
d^2 - d - 110 &= 0 \qquad \text{(subtract 104 from both sides)} \\
(d - 11)(d + 10) &= 0 \qquad \text{(factor the left-hand-side)} \\
d + 10 &= 0 \qquad \text{(divide both sides by } (d - 11)\text{)} \\
d &= -10 \qquad \text{(subtract 10 from both sides)}
\end{aligned}
$$

Therefore, my daughter's age is -10? Obviously that's a ridiculous answer, although it is a solution to the equation $(d - 3)(d + 2) = 104$, because it makes the equation true[2].

What happened? The "bad" step was when we divided both sides by $d - 11$. If one side of the equation is 0, and you divide both sides by something, you could lose solutions as we did in this example. Another way of thinking about this is: what if $d = 11$? Then dividing by $d - 11$ is like dividing by zero, a *big* no-no. In general, you want to avoid dividing by expressions that contain variables. That can be tricky, but luckily, we have a great way to deal with situations like this.

When One Side of the Equation is 0

There's a great fact we can use. If $AB = 0$, then there are exactly two possibilities – either $A = 0$ or $B = 0$. So if we reduce an equation down to a form $AB = 0$, then we can split it up into two equations – $A = 0$ and $B = 0$. Solutions to either of these simpler equations will be solutions to the main equation.

Let's solve this equation the correct way.

$$
\begin{aligned}
(d - 3)(d + 2) &= 104 \\
d^2 - d - 6 &= 104 \qquad \text{(multiply out the left-hand-side using FOIL)} \\
d^2 - d - 110 &= 0 \qquad \text{(subtract 104 from both sides)} \\
(d - 11)(d + 10) &= 0 \qquad \text{(factor the left-hand-side)}
\end{aligned}
$$

[2]Check it! Plug in $d = -10$, and you get $(-10 - 3)(-10 + 2) = (-13)(-8) = 104$

Now don't divide both sides by something. Recognize that we have a form $AB = 0$[3], so we can split it up into two equations:

$$d - 11 = 0 \qquad \text{or} \qquad d + 10 = 0.$$

Each of these simpler equations is easy to solve, and so we get two answers: $d = 11$ and $d = -10$. Therefore, my daughter's age must be 11, since -10 doesn't make sense.

Here's another example: solve the equation $x^3 - 6x = x^2$.

Start by simplifying the equation. If you don't know what else to do, get all the unknowns on one side.

$$x^3 - x^2 - 6x = 0$$

If you don't know what to do next, try factoring. This is especially good if the other side equals 0, because you can use our new technique.

$$x(x^2 - x - 6) = 0$$

Can you factor even more?

$$x(x - 3)(x + 2) = 0$$

Then use the rule. You're not limited to $AB = 0$, it works the same if you have something of the form $ABC = 0$.

$$x = 0 \qquad \text{or} \qquad x - 3 = 0 \qquad \text{or} \qquad x + 2 = 0$$

So there are three possible solutions: $x = 0, 3$, or -2.

Okay, it's about time you started solving some equations of your own. Let's just put the warning in now: check your answers! Even if you follow all the rules perfectly, you can sometimes get solutions that aren't really solutions (and this will happen in the problems below!). Some of the problems are "weird" and you'll have to figure out what's going on.

(1) $\frac{y}{3} + 4 = \frac{y}{2}$

(2) $\frac{2}{3}x + 3(x - 1) = 8$

(3) $3 + 2x = 2(\frac{3}{2} + x)$

(4) $\frac{1}{x} = \frac{3}{x} + 1$

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[3] A in this case is $d - 11$ and B is $d + 10$

(5) $\frac{3}{5}x + 8 = -x + \frac{1}{5}(2 + 8x)$

(6) $\sqrt{x} = 2 - x$

(7) $\frac{2z-1}{z+2} = \frac{4}{5}$

(8) $y = 1 + \sqrt{2 - 2y}$

(9) $(x + 2)^2 = 4$

(10) $\frac{1}{x-1} + \frac{1}{x+2} = -\frac{8}{5}$

Sometimes your equations will have more than one non-number floating around. In these cases, you'll always be solving "for" something. That is, you might have the equation $ax + b = 4$, and be asked to "solve for x." That means you should treat a and b just like numbers (constants), even though they're unknown numbers. So you would solve this equation as follows:

$$\begin{aligned} ax + b &= 4 \\ ax &= 4 - b \\ x &= \frac{4 - b}{a} \end{aligned}$$

So your solution is $x = \frac{4-b}{a}$. Don't be thrown off by more exotic notations, either. There's always just the thing you're solving for, and then everything else is a constant.

(11) Solve for x:

$$\frac{a+x}{b-x} = c$$

(12) Solve for a:

$$\frac{a+x}{b-x} = c$$

(13) Solve for θ:

$$12\theta^2 + \theta - 2 = 0$$

(14) Solve for $\frac{dy}{dx}$:

$$3y^2\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 7$$